Mathematical Excalibur

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Olympiad Corner

25th United States of America Mathematical Olympiad:

Part I (9am-noon, May 2, 1996)

Problem 1. Prove that the average of the numbers $n \sin n°$ ($n = 2, 4, 6, \ldots, 180$) is $\cot 1°$.

Problem 2. For any nonempty set S of real numbers, let $\sigma(S)$ denote the sum of the elements of S. Given a set A of n positive integers, consider the collection of all distinct sums $\sigma(S)$ as S ranges over the nonempty subsets of A. Prove that this collection of sums can be partitioned into n classes so that in each class, the ratio of the largest sum to the smallest sum does not exceed 2.

Problem 3. Let ABC be a triangle. Prove that there is a line l (in the plane of triangle ABC) such that the intersection of the interior of triangle ABC and the interior of its reflection $A'B'C'$ in l has area more than 2/3 the area of triangle ABC.

(*continued on page 4*)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is Jan 31, 1997.

For individual subscription for the remaining three issues for the 96-97 academic year, send us three stamped self-addressed envelopes. Send all correspondence to:

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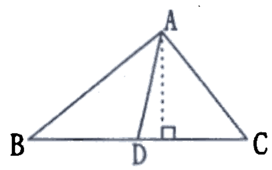
老師不教的幾何 (一)

張百康

香港中學課程不重視幾何，許多漂亮而有意義的幾何性質都被摒諸課堂以外。我在這裏嘗試逐期介紹一些重要的幾何定理，增加同學對幾何的認識。

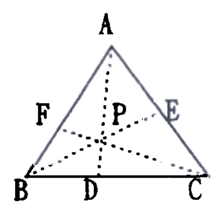
讓我們先從三角形的一個簡單而重要的性質談起：



(圖一)

圖一的兩個三角形有公共邊AD，它們是等高的，因此

$$\frac{\text{Area of }\Delta ABD}{\text{Area of }\Delta ADC}=\frac{BD}{DC}\text{。}$$



(圖二)

再看圖二，圖中的線段AD、BE和CF相交於同一點P。利用上述共邊三角形的性質可知

$$\frac{\text{Area of }\Delta ABD}{\text{Area of }\Delta ADC}=\frac{BD}{DC}=\frac{\text{Area of }\Delta PBD}{\text{Area of }\Delta PDC}$$

運用分數特性：

$$\text{若}\ \frac{a}{b}=\frac{c}{d}\text{，則}\ \frac{a}{b}=\frac{c}{d}=\frac{a-c}{b-d}$$

可得

$$\frac{BD}{DC}=\frac{\text{Area of }\Delta ABP}{\text{Area of }\Delta ACP}$$

同理

$$\frac{CE}{EA}=\frac{\text{Area of }\Delta BCP}{\text{Area of }\Delta BAP}$$

及

$$\frac{AF}{FB}=\frac{\text{Area of }\Delta ACP}{\text{Area of }\Delta BCP}$$

將上述三等式同側相乘和約項可簡化爲

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1\text{。}$$

這定理是意大利人西瓦 (Giovanni Ceva)在十七世紀時發現的，所以後人稱之爲「西瓦(Ceva)定理」。西瓦定理可用不同方法證明，但上述證法巧妙地利用了重疊圖形面積相減的手法，值得大家借鏡。

西瓦定理的逆命題是否也眞確？所謂逆命題，就是把原命題的因果對調。就西瓦定理來說，它的逆命題是:

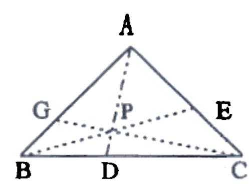
設三角形ABC的西瓦線(Cevians)[連頂點至對邊的線段]AD、BE和CF滿足條件

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1 \qquad (*)$$

則此三西瓦線共點。

此逆定理的證法特點是可借用原定理:雖然三西瓦線不一定共點，但其中兩條西瓦線必然共點。我們可設AD和BE交於P（圖三）過P作西瓦線CG，則

$$\frac{AG}{GB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1 \qquad (**)$$



(圖三)

(*continued on page 2*)

老師不教的幾何 (一)

(*continued from page 1*)

比較 (*) 和 (**) 可知

$$\frac{AF}{FB}=\frac{AG}{GB}$$

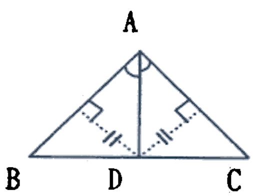
因此F和G是AB上的同一點。

西瓦定理的逆定理看似陌生，但事實上它概括了三角形的三個重要性質：三中線 (medians) 共點、三高 (altitudes) 共點、三分角線 (angle-bisectors)共點。這三個點分別名爲重心 (centroid)、垂心 (orthocentre)和內切圓心(incentre)。

三中線共點這一事實可以很容易地從西瓦定理的逆定理推出，同學們請試試。大家更可利用共邊三角形的面積關係，得出《重心把中線分成兩段長度2:1的線段》這一著名性質。

三高共點此性質可利用直角三角形邊長的三角函數關係輕易導出，也留待同學們自己試試。

三角形的分角線有一個簡單而重要的性質：



（圖四）

圖四的共邊三角形ABD和ACD的面積和邊長有下列關係

$$\frac{BD}{DC}=\frac{\text{Area of }\Delta ABD}{\text{Area of }\Delta ACD}=\frac{AB}{AC},$$

因此

$$\frac{BD}{DC}=\frac{AB}{AC}。$$

三分角線共點這一特性可借助此等式得出。

(下期續)

Fermat Point

On the outside of ΔABC draw equilateral triangles BCA', CAB' and ABC'. The three lines AA', BB' and CC' meet at a point called the Fermat Point.

Error Correcting Codes (Part I)

Tsz-Mei Ko

Suppose one would like to transmit a message, say "HELLO…", from one computer to another. One possible way is to use a table to encode the message into binary digits. Then the receiver would be able to decode the message with a similar table. One such table is the American Standard Code for Information Interchange (ASCII) shown in Figure 1. The letter H would be encoded as 1001000, the letter E would be encoded as 1000101, etc. (Figure 2).

A	1000001	S	1010011	a	1100001	s	1110011
B	1000010	T	1010100	b	1100010	t	1110100
C	1000011	U	1010101	c	1100011	u	1110101
D	1000100	V	1010110	d	1100100	v	1110110
E	1000101	W	1010111	e	1100101	w	1110111
F	1000110	X	1011000	f	1100110	x	1111000
G	1000111	Y	1011001	g	1100111	y	1111001
H	1001000	Z	1011010	h	1101000	z	1111010
I	1001001	0	0110000	i	1101001	.	0101110
J	1001010	1	0110001	j	1101010	,	0101100
K	1001011	2	0110010	k	1101011	?	0111111
L	1001100	3	0110011	l	1101100	)	0101001
M	1001101	4	0110100	m	1101101	(	1111011
N	1001110	5	0110101	n	1101110	/	0101111
O	1001111	6	0110110	o	1101111	&	0100110
P	1010000	7	0110111	p	1110000	+	0101011
Q	1010001	8	0111000	q	1110001	-	0101101
R	1010010	9	0111001	r	1110010	=	0111101

Figure 1. ASCII code

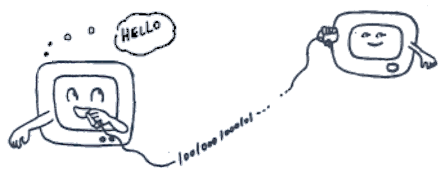


Figure 2. Two computers talking

The receiver will be able to decode the message correctly if there is no error during transmission. However, if there are transmission errors, the receiver may decode the message incorrectly. For example, the letter H (1001000) would be received as J (1001010) if there is an error at position 6.

Figure 3. Error at position 6.

One possible way to detect transmission errors is to add redundant bits, i.e., append extra bits to the original message. For an even parity code, a 0 or 1 is appended so that the total number of 1's is an even number. The letters H and E would be represented by 10010000 and 10001011 respectively. With an even parity code, the receiver can detect one transmission error, but unable to correct it. For example, if 10010000 (for the letter H) is received as 10010100, the receiver knows that there is at least one error during transmission since the received bit sequence has an odd parity, i.e., the total number of 1's is an odd number.

Figure 4. Even parity code

Is there an encoding method so that the receiver would be able to correct transmission errors? Figure 5 shows one such method by arranging the bit sequence (e.g., 1001) into a rectangular block and add parity bits to both rows and columns. For the example shown, 1001 would be encoded as 10011111 (by first appending the row parities and then the column parities). If there is an error during transmission, say at position 2, the receiver can similarly arrange the received sequence 11011111 into a rectangular block and detect that there is an error in row 1 and column 2.

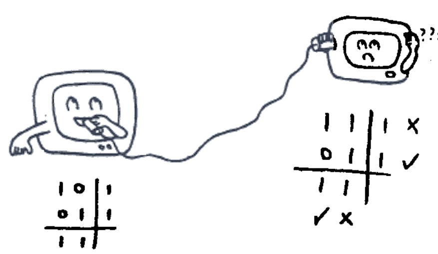


Figure 5. A code that can correct 1 error.

The above method can be used to correct one error but rather costly. For every four bits, one would need to transmit an extra four redundant bits. Is there a better way to do the encoding? In 1950, Hamming found an ingenious method to

(*continued on page 4*)

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is Jan 31, 1997.

Problem 46. For what integer a does $x^2 - x + a$ divide $x^{13} + x + 90$?

Problem 47. If x, y, z are real numbers such that $x^2 + y^2 + z^2 = 2$, then show that $x + y + z \le xyz + 2$.

Problem 48. Squares *ABDE* and *BCFG* are drawn outside of triangle *ABC*. Prove that triangle *ABC* is isosceles if *DG* is parallel to *AC*.

Problem 49. Let u_1, u_2, u_3, … be a sequence of integers such that $u_1 = 29$, $u_2 = 45$ and $u_{n+2} = u_{n+1}^2 - u_n$ for $n = 1, 2, 3, \ldots$. Show that 1996 divides infinitely many terms of this sequence. (Source: 1986 Canadian Mathematical Olympiad with modification)

Problem 50. Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle in a given direction (that is, the numbers a, b, c, d are replaced by $a - b$, $b - c$, $c - d$, $d - a$). Is it possible after 1996 such steps to have numbers a, b, c, d such that the numbers $|bc - ad|$, $|ac - bd|$, $|ab - cd|$ are primes? (Source: unused problem in the 1996 IMO.)

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Solutions

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Problem 41. Find all nonnegative integers x, y satisfying $(xy - 7)^2 = x^2 + y^2$.

Solution: Gary NG Ka Wing (STFA Leung Kau Kui College, Form 4).

Suppose x, y are nonnegative integers such that $(xy - 7)^2 = x^2 + y^2$. Then $(xy - 6)^2 + 13 = (x + y)^2$ by algebra. So

$$13 = [(x+y) + (xy-6)][(x+y) - (xy-6)].$$

Since 13 is prime, the factors on the right side can only be ± 1 or ± 13. There are four possibilities yielding $(x,y) = (0,7), (7,0), (3,4), (4,3)$.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 6), **CHENG Wing Kin** (S.K.H. Lam Woo Memorial Secondary School, Form 5), **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 6), **Yves CHEUNG Yui Ho** (S.T.F.A. Leung Kau Kui College, Form 5), **CHING Wai Hung** (S.T.F.A. Leung Kau Kui College, Form 5), **CHUI Yuk Man** (Queen Elizabeth School, Form 7), **LIU Wai Kwong** (Pui Tak Canossian College), **POON Wing Chi** (La Salle College, Form 7), **TING Kwong Chi & David GIGGS** (SKH Lam Woo Memorial Secondary School, Form 5), **YU Chun Ling** (HKU) and **YUNG Fai** (CUHK).

Problem 42. What are the possible values of $\sqrt{x^2+x+1}-\sqrt{x^2-x+1}$ as x ranges over all real numbers?

Solution: William CHEUNG Pok-man (STFA Leung Kau Kui College, Form 6).

Let $A=(x,0)$, $B=(-\frac{1}{2},\frac{\sqrt{3}}{2})$, $C=(\frac{1}{2},\frac{\sqrt{3}}{2})$. The expression $\sqrt{x^2+x+1}-\sqrt{x^2-x+1}$ is just $AB - AC$. As x ranges over all real numbers, A moves along the real axis and the triangle inequality yields

$$-1 = -BC < AB - AC < BC = 1.$$

All numbers on the intergal $(-1,1)$ are possible.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 6), **CHENG Wing Kin** (S.K.H. Lam Woo Memorial Secondary School, Form 5), **LIU Wai Kwong** (Pui Tak Canossian College), **POON Wing Chi** (La Salle College, Form 7), **YU Chun Ling** (HKU) and **YUNG Fai** (CUHK).

Problem 43. How many 3-element subsets of the set $X = \{1, 2, 3, \ldots, 20\}$ are there such that the product of the 3 numbers in the subset is divisible by 4?

Solution: CHAN Ming Chiu (La Salle College, Form 6), **CHAN Wing Sum** (HKUST), **CHENG Wing Kin** (S.K.H. Lam Woo Memorial Secondary School, Form 5), **CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 6), **William CHEUNG Pok-man** (S.T.F.A. Leung Kau Kui College, Form 6), **Yves CHEUNG Yui Ho** (S.T.F.A. Leung Kau Kui College, Form 5), **CHUI Yuk Man** (Queen Elizabeth School, Form 7), **FUNG Tak Kwan** (La Salle College, Form 7), **LEUNG Wing Lun** (STFA Leung Kau Kui College, Form 6), **LIU Wai Kwong** (Pui Tak Canossian College), **Henry NG Ka Man** (STFA Leung Kau Kui College, Form 6), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4), **POON Wing Chi** (La Salle College, Form 7), **TSANG Sai Wing** (Valtorta College, Form 6), **YU Chun Ling** (HKU), **YUEN Chu Ming** (Kiangsu-Chekiang College (Shatin), Form 6) and **YUNG Fai** (CUHK).

There are $C_3^{20} = 1140$ 3-element subsets of X. For a 3-element subset whose 3 numbers have product not divisible by 4, the numbers are either all odd (there are $C_3^{10} = 120$ such subsets) or two odd and one even, but the even one is not divisible by 4 (there are $C_2^{10} \times 5 = 225$ such subsets). So the answer to the problem is $1140 - 120 - 225 = 795$.

Problem 44. For an acute triangle *ABC*, let H be the foot of the perpendicular from A to BC. Let M, N be the feet of the perpendiculars from H to AB, AC, respectively. Define L_A to be the line through A perpendicular to MN and similarly define L_B and L_C. Show that L_A, L_B and L_C pass through a common point O. (This was an unused problem proposed by Iceland in a past IMO.)

Solution: William CHEUNG Pok-man (STFA Leung Kau Kui College, Form 6).

Let L_A intersect the circumcircle of ΔABC at A and E. Since $\angle AMH = 90° = \angle ANH$, A, M, H, N are concyclic. So $\angle MAH = \angle MNH = 90° - \angle ANM = \angle NAE = \angle CBE$. Now $\angle ABE = \angle CBE + \angle ABC = \angle MAH + \angle ABC = 90°$. So AE is a diameter of the circumcircle and

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

L_A passes through the circumcenter O. Similarly, L_B and L_C will pass through O.

Other commended solvers: **Calvin CHEUNG Cheuk Lun** (STFA Leung Kau Kui College, Form 5), **LIU Wai Kwong** (Pui Tak Canossian College), **POON Wing Chi** (La Salle College, Form 7) and **YU Chun Ling** (HKU).

Problem 45. Let $a, b, c > 0$ and abc=1. Show that

$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}\le 1$$

(This was an unused problem in IMO96.)

Solution: YUNG Fai (CUHK).

Expanding $(a^3-b^3)(a^2-b^2) \ge 0$, we get $a^5+b^5 \ge a^2b^2(a+b)$. So using this and $abc = 1$, we get

$$\frac{ab}{a^5+b^5+ab} < \frac{ab}{\qquad} \times \frac{c^2}{c^2}$$

$$= \frac{\qquad}{a+b+c}.$$

Adding 3 such inequalities, we get the desired inequality. In fact, equality can occur if and only if $a = b = c = 1$.

Other commended solvers: **POON Wing Chi** (La Salle College, Form 7) and **YU Chun Ling** (HKU).

Olympiad Corner

(*continued from page 1*)

Part II (1pm-4pm, May 2, 1996)

Problem 4. An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a *binary sequence of length n*. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

Problem 5. Triangle ABC has the following property: there is an interior point P such that $\angle PAB = 10°$, $\angle PBA = 20°$, $\angle PCA = 30°$, $\angle PAC = 40°$. Prove that triangle ABC is isosceles.

Problem 6. Determine (with proof) whether there is a subset X of the integers with the following property: for any integer n there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

Error Correcting Codes (Part I)

(*continued from page 2*)

add the redundancy. To encode a four-bit sequence $p_1p_2p_3p_4$ (say 1001), one would first draw three intersecting circles A, B, C and put the information bits p_1, p_2, p_3, p_4 into the four overlapping regions $A \cap B$, $A \cap C$, $B \cap C$ and $A \cap B \cap C$ (Figure 6). Then three parity bits p_5, p_6, p_7 are generated so that the total number of 1's in each circle is an even number. For the example shown, 1001 would be encoded as 1001001.

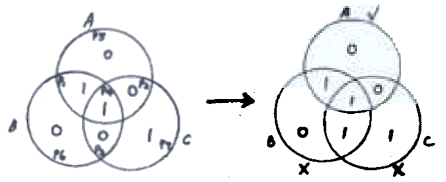

Figure 6. Hamming code

If there is one error during transmission, say 1001001 received as 1011001, the receiver can check the parities of the three circles to find that the error is in circles B and C but not in A. This (7,4) Hamming code (the notation (7,4) means that every 4 information bits are encoded as a 7 bit sequence) can be generalized. For example, one may draw 4 intersecting spheres in a three-dimensional space to obtain a (15,11) Hamming code. Hamming has also proved that his coding method is optimum for single error correction.

(... to be continued)